Stochastic resonance in the Brusselator model

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Using the Brusselator model, we show that in a simple dynamical system small noise can be converted into stochastic spikewise oscillations of huge amplitude (bursting noises) in the vicinity of a Hopf bifurcation. Small periodic signals with amplitude several times less than the noise intensity transform these stochastic oscillations into quasiperiodic large-amplitude spikewise oscillations or small-amplitude quasiharmonic oscillations, depending on the signal form.

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There are nonlinear systems that possess a large noise-dependent susceptibility [1–5]. In these systems phase transitions can occur under the action of small noises in them [6]. They can greatly amplify weak signals in the presence of the noise, and the signal-to-noise ratio in them can increase with noise. The supersensitivity to noises and small signals is observed in different kinds of systems and is called stochastic resonance (SR) [1–5]. In particular, SR takes place in bistable systems where it is characterized by random transitions from the one stable state to the other stimulated by both noise and the signal [1–5].

In this paper we show that SR can occur in simple monostable dynamical systems in the vicinity of a Hopf bifurcation. We use the Brusselator model as an example.

The Brusselator model is a widely used model of autocatalytic chemical reactions. A scheme of this model is [7]

$$A \rightarrow X, 2X+Y \rightarrow 3X, B+X \rightarrow Y+C, X \rightarrow E, \tag{1}$$

where A and B are the initial chemical substances, C and E are the final chemical substances, and X and Y are the intermediate chemical substances. The second of these reactions is autocatalytic and describes self-production of the substance X called the activator. The process of self-reproduction is controlled by substance Y called the inhibitor. In the simplest case the reactions (1) are described by the following equations [7,8]:

$$\tau_\theta \frac{d\theta}{dt} = 1-(A+1)\theta + \theta^2 \eta, \tag{2}$$

$$\tau_\eta \frac{d\eta}{dt} = A\theta - \theta^2 \eta, \tag{3}$$

where θ and η are concentrations of the activator and the inhibitor, respectively; τ_x and τ_y are time scales of the θ and η; A is a control parameter. Equations (2) and (3) were analyzed in [7–10]. The nullcline of Eq. (2) has Λ-like shape [thin curve Λ in Fig. 1(a)] and intersects the nullcline of Eq. (3) at the point $\theta=\theta^0=1$ and $\eta=\eta^0=A$. This point determines the only equilibrium state. This state is unstable when $A>A_\omega=1+\alpha$ with respect to the perturbations with the frequency $\omega_0=\alpha^{-1/2}$, where $\alpha=\tau_\theta/\tau_\eta$.

Let us analyze in more detail the small-amplitude quasiharmonic oscillations for A close to A_ω when $\alpha \ll 1$. We seek

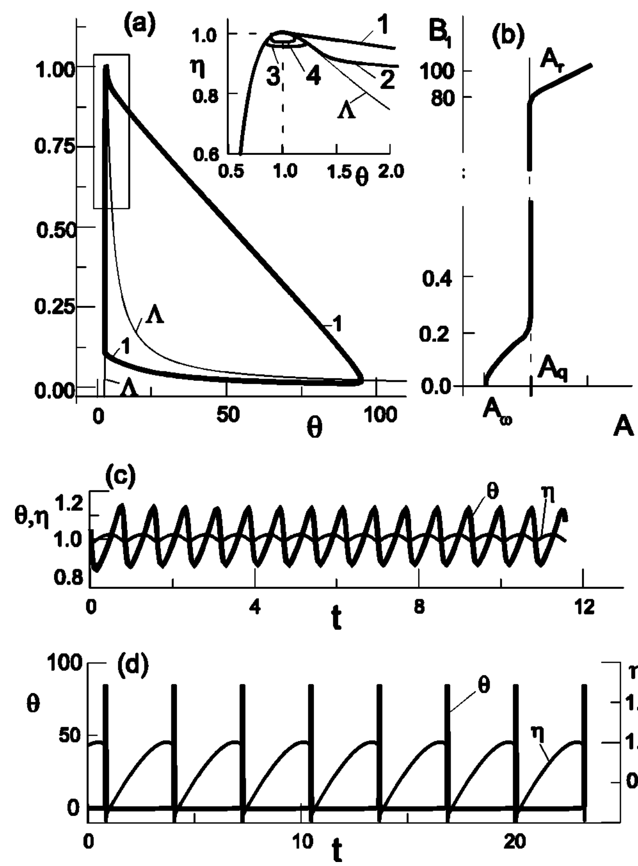

FIG. 1. Quasiharmonic and spikewise relaxation oscillations: the stable limit cycles (a), the bifurcation diagram (b), the dependence of the amplitudes of the oscillations B_1 on the control parameter A; the forms of the quasiharmonic oscillations (c) and the relaxation oscillations (d). From the numerical solution of Eqs. (2) and (3) with $\alpha=0.01$ and $A=A_q+0.1\alpha=1.016$ for (a) and (d). In (a) the thin curve Λ shows the nullcline of Eq. (2). The upper inset at the right of (a) shows the limit cycle near the unstable equilibrium state $\theta=\theta_0=1$ and $\eta=\eta_0=A$ for different values of $\Delta A = A-A_q$, where $A_q=A_r=1+1{,}5\alpha$ and $\Delta A=0.1\alpha$ for curve 1, $\Delta A=0.05\alpha$ for curve 2, $\Delta A=-0.05$ for curve 3, $\Delta A=-0.1\alpha$ for curve 4. The unit of time is τ_η.

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the amplitude of the oscillations as a series $\tilde{\theta} = (\theta - \theta_0) = \Sigma B_n \exp(in\omega t)$. Substituting $\tilde{\theta}$ into Eqs. (2) and (3) and using the Bogolyubov-Mitropolskii method [11] we find that the amplitude B_1 is determined by the equation

$$B_1^2 = \frac{\alpha(A - A_\omega)}{\alpha - 2(A - A_\omega)}. \tag{4}$$

From Eq. (4) follows that $dB_1/dA = \infty$ at two points [Fig. 1(b)]: $A = A_\omega$ and $A = A_q = 1 + \frac{3}{2}\alpha = A_\omega + \frac{1}{2}\alpha$. Note that the difference $(A_q - A_\omega)$ is very small when $\alpha \ll 1$. This means that the small-amplitude quasiharmonic oscillations [Fig. 1(c)] occur only for A very close to A_ω. Their amplitude increases sharply when $A \to A_q$ and they transform abruptly into periodical relaxation oscillations at $A > A_q$. These relaxation oscillations have the form of spikes [Fig. 1(d)]. Their amplitude is proportional to $\alpha^{-1} \gg 1$ and the period is determined by value τ_η [7,8,12,13]. The limit cycle corresponding to the spikewise oscillations is shown by the thick curve 1 in Fig. 1(a).

When A is decreased the relaxation oscillations abruptly disappear at $A = A_r$. The general formula for A_r was obtained by Baer and Erneux [14]. From this formula it follows that $A_r = A_\omega + \frac{1}{2}\alpha$ for the Brusselator model up to second order in α. Thus $A_r = A_q > A_\omega$ and the bifurcation diagram of the Brusselator model has the form shown in Fig. 1(b), i.e., the Brusselator model is dynamically monostable for all values of A.

From Fig. 1(b) one can see that the small-amplitude quasiharmonic and the large-amplitude relaxation oscillations transform abruptly one into another in a very small region near $A = A_q = A_r$. It means that small fluctuations δA of the parameter A [Fig. 2(a)] near $A = A_q$ can transform into huge (bursting) noises related to the random transitions between the quasiharmonic and the relaxation oscillations [Fig. 2(b)].

The fluctuations $\delta A = [A(t) - A_0]$, where $A_0 \simeq A_q$, have been simulated as a random pulse sequence. The pulse amplitude and duration had a Gaussian distribution with the dispersion σ^2. For stochastic integration we have used the Euler method to avoid problems with correlations to the noise. We have chosen an integration step which is at least an order of magnitude smaller than the mean duration of the random pulses and α. We found that the same results are obtained by using the fourth-order Runge-Kutta method.

We carried out the numerical analysis of Eqs. (2) and (3) with $\alpha \ll 1$ and found the following results. The quasiharmonic oscillations [Fig. 1(c)] arise spontaneously when A_0 satisfies the condition $A_\omega < A_0 < A_q$ and the fluctuations δA are very small. When the noise intensity [Fig. 2(a)] σ exceeds some critical value (0.006 for $\alpha = 0.01$ and $A_0 = 1.0145$) these oscillations transform into a stochastic sequence of spikes of different amplitudes and duration which alternate with quasiharmonic small-amplitude oscillations [Fig. 2(b)]. When the noise intensity increases tandems of several spikes appear. They are separated by the intervals of the order of τ_η and their amplitudes are close to each other. At relatively high values of σ ($\sigma > 0.02$) the stochastic oscillations transform into the quasiperiodic relaxation spikewise oscillations (Fig. 1(d)]. Thus the stochastic oscillations

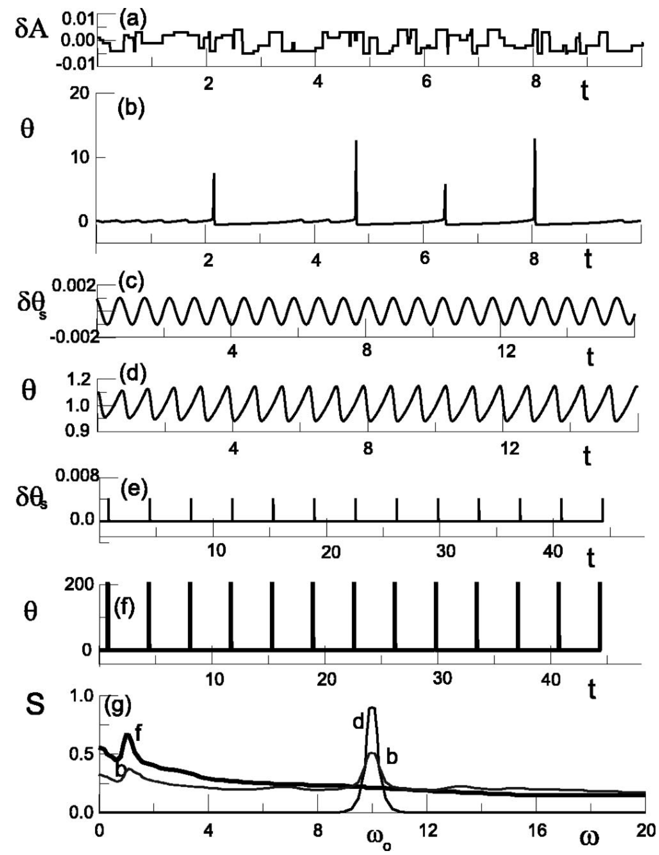

FIG. 2. Oscillations in the presence of the noise and the small signals: fluctuations of the control parameter A (a), stochastic spikewise oscillations (b), small sinusoidal signal with frequency $\omega \simeq \omega_0$ (c), the corresponding quasiharmonic oscillations (d), small periodic spikewise signal (e), the corresponding quasiperiodic spikewise relaxation oscillations (f), the spectra corresponding to these oscillations (g). From the numerical solution of Eqs. (2) and (3) with $\alpha = 0.01$ and $A = A_q - 0.05\alpha$, where $A_q = A_r = 1 + 1{,}5\alpha$ and the noise intensity $\sigma = 0.01$. The curves b, d, and f, in (g) correspond to the oscillations shown in (b), (d), and (f). The unit of time is τ_η.

occur in a certain range of σ ($0.006 < \sigma < 0.02$ for $\alpha = 0.01$ and $A = 1.0145$). Their spectrum is presented by curve b in Fig. 2(g). We have calculated the spectrum using the standard Fast Fourier Transform (FFT), method and have normalized it to the square root of the oscillation power.

We have found that the stochastic oscillations are supersensitive to very small external signals. These signals have been modeled by an additional term $b(t)$ in the right-hand side of Eq. (2). It means that the concentration of initial substance B varies with time. Our simulations show that the stochastic spikewise oscillations [Fig. 2(b)] transform into small-amplitude quasiharmonic oscillations [Fig. 2(d)] when the signal has periodic form; in particular, when $b(t) = b_0 \cos(\omega t)$ with the frequency $\omega \approx \omega_0$ [Fig. 2(c)]. The spectrum of the quasiharmonic oscillations is shown by curve d in Fig. 2(g). This effect takes place when $b_0 > 0.2\sigma$, i.e., even when the signal amplitude is about five times less than the noise intensity σ.

When the signal has the form of the small-amplitude spikes with the period of order τ_η [Fig. 2(e)] the stochastic oscillations [Fig. 2(b)] transform into quasiperiodic relaxation spikewise oscillations [Fig. 2(f)]. Their spectrum is shown by curve *f* in Fig. 2(g). This effect takes place when the spike amplitude is about five times less than the noise intensity σ. Thus, small periodic spikewise signals are amplified more than 10^4 times in the presence of the fluctuations of *A*. Their amplification does not occur without the fluctuations.

Our calculations also show that the signal-to-noise ratio (SNR) reaches a maximum at $\sigma \simeq 0.01 = \alpha$. This value lies approximately in the middle of the range $0.006 < \sigma < 0.02$ of the existence of the stochastic oscillations. This result is a general property of SR [1–5].

In conclusion we note that the supersensitivity of the stochastic relaxation oscillations to very small signals is a rather universal phenomenon. Our study shows that the bursting noises and the stochastic resonance take place in simple dynamical systems where relaxation oscillations arise abruptly and the value of the control parameter lies in the vicinity of a Hopf bifurcation. In particular, we numerically confirmed that such supersensitivity to the small noise and the small signals occurs in nonequilibrium photogenerated electron-hole plasma [15], the biochemical Gierer-Meinhardt model of morphogenesis [12], and the Hindmarsh-Rose model of excitable neurons [16]. Equations for the models are given accordingly in [9,17,18].

[1] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).

[2] M.I. Dykman, D.G. Luchinsky, R. Mannella, P.V.E. McClintock, N.D. Stein, and N.G. Stocks, Nuovo Cimento D **18**, 661 (1995).

[3] V.S. Anischenko, A.B. Neiman, F. Moss, and L. Schimansky-Geier, Usp. Fiz. Nauk **179**, 7 (1999) [Sov. Phys. Usp. **179**, 7 (1999)].

[4] K. Wiesenfeld and F. Moss, Nature (London) **373**, 33 (1995); L. Gammaitoni, Phys. Rev. E **52**, 4691 (1995).

[5] A. Longtin, Phys. Rev. E **55**, 868 (1997).

[6] W. Horsthemke and R. Lefever, *Noise-Induced Phase Transitions* (Springer, Berlin, 1984).

[7] G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems* (Wiley, New York, 1977).

[8] W. Ebeling, *Strukturbildung Bei Irreversiblen Prozessen* (Teunber, Leipzig, 1976); A. Engel, W. Ebeling, R. Feistel, and L. Schimansky-Geier, *Self-organization by Nonlinear Irreversible Processes*, edited by W. Ebeling and H. Ulbricht (Springer, Berlin, 1986).

[9] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984).

[10] B.S. Kerner and V.V. Osipov, *Autosolitons* (Kluwer, Dordrecht, 1994); Usp. Fiz. Nauk **157**, 201 (1989) [Sov. Phys. Usp. **32**, 101 (1989)].

[11] N.N. Bogoliubov and Y.A. Mitropolsky, *Asymptotic Methods in the Theory of Non-Linear Oscillations* (Gordon and Breach, New York, 1961).

[12] V.V. Osipov and E.V. Ponizovskaya, J. Commun. Technol. Electron. **43**, 198 (1998); **43**, 682 (1998).

[13] I.A. Lubashevski, C.B. Muratov, and V.V. Osipov, Physica D (to be published).

[14] S.M. Baer and T. Erneux, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **46**, 721 (1986); **52**, 1751 (1992).

[15] V.V. Osipov and E.V. Ponizovskaya, Pis'ma Zh. Éksp. Teor. Fiz. **70**, 422 (1999) [JETP Lett. **70**, 422 (1999)].

[16] V.V. Osipov and E.V. Ponizovskaya, Phys. Lett. A **238**, 369 (1998).

[17] F. Gierer and H. Meinhardt, Kybernetik **12**, 30 (1972).

[18] J.L. Hindmarsh and R.M. Rose, Proc. R. Soc. London, Ser. B **221**, 87 (1984).